Relativity, nonextensivity, and extended power law distributions

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(Received 20 June 2005; published 11 November 2005)

A proof of the relativistic H theorem by including nonextensive effects is given. As it happens in the nonrelativistic limit, the molecular chaos hypothesis advanced by Boltzmann does not remain valid, and the second law of thermodynamics combined with a duality transformation implies that the q parameter lies on the interval [0,2]. It is also proven that the collisional equilibrium states (null entropy source term) are described by the relativistic q power law extension of the exponential Juttner distribution which reduces, in the nonrelativistic domain, to the Tsallis power law function. As a simple illustration of the basic approach, we derive the relativistic nonextensive equilibrium distribution for a dilute charged gas under the action of an electromagnetic field $F^{\mu\nu}$. Such results reduce to the standard ones in the extensive limit, thereby showing that the nonextensive entropic framework can be harmonized with the space-time ideas contained in the special relativity theory.

DOI: 10.1103/PhysRevE.72.057101 PACS number(s): 05.90.+m, 05.20.−y, 03.30.+p, 05.70.Ln

In the past few years, a great deal of attention has been paid to the nonextensive Tsallis entropy both from theoretical and observational viewpoints [1–4]. Recent applications of the nonextensive entropy to an increasing number of physical problems are beginning to provide a more definite picture on the kind of scenarios where this Tsallis formalism proves to be extremely useful [3–13].

At present, self-gravitating systems and plasma physics offer the best framework for searching to nonextensive effects. The first one is characterized by very strange kinetic and thermal properties (see [5] for recent publications on this topic). Actually, collisionless stellar systems, such as galaxies, are endowed with negative specific heat, and the simplest density profiles based on the Maxwellian distribution lead to infinite mass (the so-called singular isothermal sphere). In the case of plasmas, Boghosian's treatment for a two-dimensional pure electron plasma yielded the first experimental confirmation of the Tsallis theory [6], whereas experiments related to dispersion relations for electrostatic plane-wave propagation also points to a class of power law Tsallis velocity distributions [7]. In reality, it is now widely believed that the nonequilibrium properties of such systems away from Boltzmann-Gibbs state are not completely understood [8]. This nonextensive statistical formalism also proved to be a useful construct for the analysis of many interesting properties of linear and nonlinear Fokker-Planck equations [9].

On the other hand, most of the observational or experimental evidence supporting the Tsallis proposal are related to the power-law velocity distribution associated with the Tsallis thermostatistical description of the classical N body problem [10]. For a dilute gas of massive point particles, the nonextensive effects are simply parametrized by the local entropy density formula

$$S_q = - k_B \int f^q \ln_q f d^3p, \tag{1}$$

where k_B is the Boltzmann constant, f is the distribution function, q is the nonextensive parameter, and the q-logarithmic function is defined by

$$\ln_q f = (1-q)^{-1}(f^{1-q} - 1) \quad (f > 0) \tag{2}$$

which recovers the standard Boltzmann-Gibbs entropy $S=-k_B \int f \ln f d^3p$ in the limit $q \rightarrow 1$. In the nonrelativistic limit, the time evolution of S_q was analyzed with emphasis on the Liouville and Fokker-Planck equations [11] as well as through a nonextensive generalization of the nonrelativistic Boltzmann H theorem [12,13].

As recognized by Lima *et al.* [12] (hereafter referred to as Paper I), the attempts for extending the Boltzmann kinetic theory by including nonextensive effects, which basically means a q transport equation and the associated H theorem, required a departure from the celebrated molecular chaos hypothesis advanced by Boltzmann. In this connection, it is worth noting that the q Boltzmann equation of Paper I differs from the one proposed by Kaniadakis [14], an approach based on the kinetic interaction principle, only by the assumed form of the collision integral.

Theoretically, beyond the applications closely related to the nontrivial solutions of the nonrelativistic q transport equation and the associated transport coefficients [7], it is clearly necessary to go one step further, by extending the proof of the H theorem to the relativistic and quantum domains. The basic reasons are very well known, and have partially guided the development of modern physics [15]. Actually, in the case of the Maxwell-Boltzmann distribution, this basic program has already been performed in detail to special relativity [16], quantum theory [17], as well as by including the gravitational interaction to the context of general relativity theory [18]. In particular, the collisional equilibrium of a relativistic gas of massive point particles is described by the Juttner distribution function which contains the number density, the temperature, and the local four-momentum as free parameters [15,16].

In this Brief Report, the nonrelativistic q Boltzmann equation and the H theorem discussed in Paper I are extended to the special relativistic domain through a manifestly covariant approach. As we shall see, the whole argument follows from

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a direct generalization of the molecular chaos hypothesis and the expression of the four-entropy flux in the spirit of the nonextensive Tsallis prescription. The leitmotiv of this Brief Report is to show that the kinetic nonextensive approach can also be harmonized with the space-time ideas contained in the special relativity theory.

To begin with, we recall that the proof of the standard relativistic H theorem is also based on the molecular chaos hypothesis (Stosszahlansatz), i.e., the assumption that any two colliding particles are uncorrelated. This means that the two-point correlation function of the colliding particles can be factorized

$$f(x,p,p_1) = f(x,p)f(x,p_1), \tag{3}$$

or, equivalently,

$$\ln f(x,p,p_1) = \ln f(x,p) + \ln f(x,p_1), \tag{4}$$

where the particles have four-momentum $p \equiv p^{\mu} = (E/c,\mathbf{p})$ in each point $x \equiv x^{\mu} = (ct,\mathbf{r})$ of the space-time, with their energy satisfying $E/c = \sqrt{\mathbf{p}^2 + m^2c^2}$ (in the above expressions, p and p_1 are the four-momenta just before collision). In what follows, we show that the relativistic nonextensive entropic measure is consistent with a slight departing from "Stosszahlansatz" (molecular chaos) when exact correlations are introduced. Operationally, this means that one must replace the logarithm functions appearing in Eq. (4) by their nonextensive counterpart which are represented by the q logarithmic (power laws) defined by Eq. (2). It should be recalled that the validity of the chaos molecular hypothesis still remains as a very controversial issue [19]. Probably, the unique consensus is that it is by no means a consequence of the laws of mechanics, and, as shown in Paper I, the Stosszahlansatz is not responsible by the irreversible content of the Boltzmann approach.

Let us now consider a relativistic rarified gas containing N-point particles of mass m enclosed in a volume V, and under the action of an external four-force field F^{μ}. From a kinetic viewpoint, the states of the gas must be characterized by a Lorentz invariant one-particle distribution function $f(x,p)$. By definition, the quantity $f(x,p)d^3x d^3p$ gives, at each time t, the number of particles in the volume element $d^3x d^3p$ around the particles space-time position x and momentum $\mathbf{p}$. By taking into account the nonrelativistic treatment (see [12]), one may assume that the temporal evolution of the relativistic distribution function $f(x,p)$ is driven by the following q transport equation:

$$p^{\mu}\partial_{\mu} f + mF^{\mu}\frac{\partial f}{\partial p^{\mu}} = C_q(f), \tag{5}$$

where the index μ take the four values 0,1,2,3, while $\partial_{\mu} = (c^{-1}\partial_t, \nabla)$ indicates differentiation with respect to time and space coordinates, respectively, and C_q denotes the relativistic q-collisional term. Note that the left-hand side of (5) is just the total derivative of the distribution function or the "streaming term." This means that the nonextensive effects can be manifested only through the collisional term which is a local slowly varying function of $f(x,p)$. The collision integral $C_q(f)$ must be consistent with the energy, momentum, and the particle number conservation laws, and its specific structure must be such that the standard result is recovered in the limit $q \to 1$. At this point, it is interesting to compare the approach developed here which is based on Eq. (5) with the one proposed by Lavagno [20]. In the latter work, all the nonextensive effects are quantified by assuming a modified Boltzmann equation to the quantity f^q [see Eq. (13) of the Lavagno [20]]. In particular, this means that such theories must lead to different predictions of the physical quantities, as for instance, the expressions for the transport coefficients.

Now, since $C_q(f)$ leads to a non-negative local q-entropy source, that is, $\tau_q(x) \equiv \partial_{\mu} S_q^{\mu}$, where S^{μ} is the four-entropy flux (an identically vanishing quantity for equilibrium states), its general form reads

$$C_q(f) = \frac{c}{2}\int F\sigma R_q(f,f')\frac{d^3p_1}{E_1}d\Omega, \tag{6}$$

where $d\Omega$ is an element of the collision solid angle, the scalar F is the invariant flux, which is equal to $F = \sqrt{(p_{\mu}p_1^{\mu})^2 - m^4c^4}$, and σ is the differential cross section of the collision $p + p_1 \to p' + p_1'$ (see Ref. [15] for more details). All quantities are defined in the center-of-mass system of the colliding particles. As a point of fact, relativity enters only in the definition of F, and implicitly through the differential cross section σ. The quantity $R_q(f,f')$ is a difference of two correlation functions which are assumed to satisfy a q generalized form of the molecular chaos hypothesis expressed as [12]

$$R_q(f,f') = e_q(f'^{\,q-1}\ln_q f' + f_1'^{\,q-1}\ln_q f_1') - e_q(f^{q-1}\ln_q f + f_1^{q-1}\ln_q f_1), \tag{7}$$

where primes refer to the distribution function after collision. Note that in the limit $q \to 1$, the above expression reduces to $R_1 = f'f_1' - ff_1$, thereby showing that the molecular chaos hypothesis is readily recovered. Similarly, the nonextensive four-entropy flux reads

$$S_q^{\mu} = -k_B c^2 \int p^{\mu} f^q \ln_q f \frac{d^3p}{E}, \tag{8}$$

and as should be expected, $c^{-1}S_q^0$ is just the local Tsallis' entropy density as given by (1). Now, in order to obtain the source term, we first take the four-divergence of S_q^{μ}

$$\partial_{\mu} S_q^{\mu} \equiv \tau_q = -k_B c^2 \int (qf^{q-1}\ln_q f + 1)p^{\mu}\partial_{\mu} f \frac{d^3p}{E}, \tag{9}$$

and combining with the nonextensive relativistic Boltzmann equation (5), one may rewrite the above expression in the following form:

$$\tau_q = -\frac{k_B c^3}{2}\int F\sigma(qf^{q-1}\ln_q f + 1)R_q \frac{d^3p}{E}\frac{d^3p_1}{E_1}d\Omega. \tag{10}$$

At this point, it is convenient to rewrite τ_q in a more symmetrical form by using some elementary symmetry operations which also take into account the inverse collisions. First we notice that by interchanging p and p_1 the value of the integral is preserved. This happens because the scattering cross section and the magnitude of the flux are invariants [15]. In addition, the value of τ_q is not altered if we integrate with respect to the variables p' and p'_1. Actually, although changing the sign of R_q in this step (inverse collision), the quantity $d^3pd^3p_1/p^0p_1^0$ is also a collisional invariant [15]. Finally, as we have done in Paper I, we apply a "duality" transformation (see the discussion below and Ref. [21]) of the form $f^{q-1}\ln_q f = \ln_{q*} f$, where the new nonextensive parameter is related to the old one by $q* = 2-q$. As one may note, such considerations imply that the q entropy source term can be written as

$$\tau_q(x) = \frac{qk_Bc^3}{8}\int F\sigma(\ln_{q^*} f' + \ln_{q^*} f'_1 - \ln_{q^*} f - \ln_{q^*} f_1) \times[(e_q(\ln_{q^*} f' + \ln_{q^*} f'_1) - e_q(\ln_{q^*} f - \ln_{q^*} f_1))] \times\frac{d^3p}{E}\frac{d^3p_1}{E_1}d\Omega. \quad (11)$$

This is our main result, and the reader should compare it with the nonrelativistic expression deduced in Paper I. As widely known, the irreversible nature of thermodynamics emerging from molecular collisions is recovered if the above quantity is positive definite. In the present case, such a condition can be guaranteed in two steps. First, we notice that the integrand of

$$(\ln_{q^*} f' + \ln_{q^*} f'_1 - \ln_{q^*} f - \ln_{q^*} f_1)[e_q(\ln_{q^*} f' + \ln_{q^*} f'_1) - e_q(\ln_{q^*} f - \ln_{q^*} f_1)] \quad (12)$$

is always positive for any pair of distributions (f,f_1) and (f',f'_1). This means that the sign of the four-entropy source is now completely determined by the sign of the nonextensive parameter. Therefore, if the second law is to be obeyed [15,22], the values of this parameter must be restricted to $q \geqslant 0$. In other words, when $q<0$, the relativistic q entropy source of a given volume element decreases in the course of time. Note that the boarder case $(q=0)$ seems to be physically meaningless, since the entropy is constant regardless of the solution obtained from the transport equation with a nonnull collision integral. In this concern, one may ask if the q parameter is limited from above. As one may note, repeating all the calculations present until now with a duality transformation, i.e., by taking $q* = 2-q$ in the Tsallis entropy, Eq. (1), it is easy to conclude that $q* > 0$. Therefore, the duality transformation together with the relativistic H theorem imply that q is constrained on the interval [0,2] (as pointed out by Karlin *et al.* [21]; such a result is also valid in the nonrelativistic theory [12]). In particular, this means that the upper bound of q, i.e., $q<2$ is not a purely quantum mechanics restriction, as recently claimed in the literature [23].

In order to complete the proof of the theorem, let us now derive the nonextensive Juttner distribution. Such a function is the relativistic version of the q power Tsallis distribution [10,12], and must be obtained as a natural consequence of the relativistic H theorem. As happens in the classical case, $\tau_q=0$ is a necessary and sufficient condition for local and global equilibrium. Since the integrand appearing on the expression of τ_q must be positive definite, this occurs if and only if

$$\ln_{q^*} f' + \ln_{q^*} f'_1 = \ln_{q^*} f + \ln_{q^*} f_1, \quad (13)$$

where the four-momenta are connected through a conservation law $(p^\mu + p_1^\mu = p'^\mu + p_1'^\mu)$ which is valid for any binary collision. Therefore, the above sum of q logarithms remains constant during a collision. It is a summational invariant. In the relativistic case, the most general collision invariant is a linear combination of a constant plus the four-momentum p^μ [15]. Consequently, we must have

$$\ln_{q^*} f^0(x,p) = \alpha(x) + \beta_\mu p^\mu, \quad (14)$$

where $\alpha(x)$ is a scalar, β_μ a four-vector, and p^μ is the four-momentum. After simple algebra, we may rewrite (14) as a relativistic nonextensive distribution

$$f^0(x,p) = \{1-(1-q)[\alpha(x) + \beta_\mu p^\mu]\}^{1/1-q}, \quad (15)$$

with arbitrary space and time-dependent parameters $\alpha(x)$ and $\beta_\mu(x)$. The above expression is the relativistic version of the q Tsallis distribution [12]. The function $f^0(x,p)$ is the most general expression which leads to a vanishing collision term and entropy production, and reduces to Juttner distribution in the limit $q \to 1$. However, it is not true in general that $f^0(x,p)$ is a solution of the transport equation. This happens only if f^0 also makes the left-hand side of the transport equation (5) to be identically null. Nevertheless, since (15) is a power law, the transport equation implies that the parameters $\alpha(x)$ and $\beta_\mu(x)$ must only satisfy the constraint equation

$$p^\mu\partial_\mu\alpha(x) + p^\mu p^\nu\partial_\mu\beta_\nu(x) + m\beta_\mu(x)F^\mu(x,p) = 0. \quad (16)$$

The nonextensive distribution of the form (15), with the specific parameters obeying the above equation, describes the relativistic (nonextensive) local equilibrium states.

For illustration purposes, let us now consider a relativistic gas under the action of the Lorentz four-force $F^\mu(x,p) = -(Q/mc)F^{\mu\nu}(x)p_\nu$, where Q is the charge of the particles and $F^{\mu\nu}$ is the Maxwell electromagnetic tensor. Following standard lines, it is easy to show that the local equilibrium function in the presence of an external electromagnetic field reads

$$f(x,p) = \left[1-(1-q)\left(\frac{\mu - [p^\mu + c^{-1}QA^\mu(x)]U_\mu}{k_BT}\right)\right]^{1/1-q}, \quad (17)$$

where U_μ is the mean four-velocity of the gas, $T(x)$ is the temperature field, μ is the Gibbs function per particles, and $A^\mu(x)$ the four-potential. Note that the above expression in the limit $q \to 1$ reduces to the well-known expression [15,24]

$$f(x,p) = \exp\left(\frac{\mu - [p^\mu + c^{-1}QA^\mu(x)]U_\mu}{k_BT}\right). \quad (18)$$

In summary, we have proposed a q generalization of the relativistic Boltzmann's equation and the associated H theorem along the lines of the Tsallis' nonextensive kinetic theory. We have found that the nonextensive ideas can be consistently extended in order to incorporate the space-time concepts of the special relativity. In addition, since the basic results were derived in a manifestly covariant way, their generalization to the general relativistic framework can be readily accomplished.

It is worth noting that the relativistic counterpart of the *H* theorem constrains the physically allowed values for the *q* parameter (as it occurs in the Newtonian regime), and its proof also does not require the Stosszahlansatz (molecular chaos) Boltzmann assumption. For the reasons discussed before, the *q* nonextensive contributions must appear explicitly only in the collisional term of the *q* transport equation, and, as such, the approach followed here differs profoundly from another attempt to generalize the Boltzmann equation within the spirit of the Tsallis' framework [25,20]. As should be expected, the relativistic class of *q* distributions reduce to the standard Juttner result in the extensive limit $q=1$. However, different from the extensive Maxwell-Boltzmann-Juttner approach, correlations are extremely relevant in the nonextensive context (see also Paper I), and, more importantly, the corresponding modifications in the collisional term are consistent with the standard laws of microscopic dynamics. As we have shown, such correlations are exactly described and form the physical basis of the nonextensive *H* theorem either for the relativistic and nonrelativistic regimes.

It should be stressed that the combination of the relativistic *H* theorem and duality transformation [21] restricted the *q* parameter on the range [0,2], which is exactly the same result of the nonrelativistic quantum domain [23] and of the consistent framework for generalized statistical mechanics [26]. It should be noted, however, that the allowed range of *q* may be even smaller if one takes into account the finite normalization condition and the negativeness of heat capacity. In the nonrelativistic regime, for instance, it has been shown that *q* must be smaller than 5/3 [5,27,28]. Finally, two points to be noted here and explored in the near future are the possible connection between the relativistic nonextensive function and the κ distributions [29]; and the search to the expressions of the *q* relativistic transport coefficients [30].

The authors are grateful to Angel Plastino and Jailson Alcaniz for helpful discussions. This work was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil).

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